Single Snapshot Spatial Smoothing With Improved Effective Array Aperture

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Abstract—Spatial smoothing is a widely used preprocessing scheme for direction-of-arrival (DOA) estimation of more than one source from a single snapshot, although the effective array aperture gets reduced by this process. In this paper we propose a preprocessing scheme applicable for DOA estimation algorithms that exploit the shift invariance property of the array steering matrix and call it spatial smoothing with improved aperture (SSIA). SSIA, when applied to a noise corrupted data vector, improves the effective array aperture significantly as opposed to conventional spatial smoothing. Simulations confirm the significant performance gain provided by SSIA in conjunction with Unitary ESPRIT.

Index Terms— Centro-symmetric array, direction of arrival estimation, ESPRIT, spatial smoothing, spatial smoothing with improved aperture (SSIA).

I. INTRODUCTION

ESTIMATING the directions-of-arrival (DOAs) of several sources impinging on an array of sensors has been studied for a long time. Such a problem has wide applications in radar, mobile communication, sonar, and seismology. ESPRIT [1] is one such widely used subspace-based high-resolution technique for estimating the DOAs. Unitary ESPRIT [2] exploits the centro-symmetry of a sensor array and enables computations in the real domain, thereby substantially reducing the complexity of ESPRIT.

Spatial smoothing is a preprocessing scheme that is applied to circumvent the problems encountered in DOA estimation of fully correlated signals [3], [4] impinging on a centro-symmetric array. Spatial smoothing is also employed when the DOAs of multiple sources are to be estimated from a single snapshot or observation. A disadvantage of this preprocessing scheme is that it reduces the effective array aperture. The estimates of the DOAs can be improved by incorporating forward-backward averaging [5] that effectively doubles the number of data samples after spatial smoothing.

In this paper, we propose spatial smoothing with improved aperture (SSIA), a preprocessing scheme, that can be employed when the DOAs of multiple sources are to be estimated from a single snapshot using techniques that exploit the shift invariance property of the array steering matrix such as Matrix-Pencil, ESPRIT, or Unitary ESPRIT. The effective array aperture in SSIA is significantly higher than the effective array aperture of a spatially smooth data matrix by doubling the rows of the array steering matrix. Hence, the performance of SSIA is better than existing techniques that incorporate forward-backward averaging.

Notation

We denote \( \mathbf{P}_p \) as a \( p \times p \) exchange matrix with ones on its antidiagonal and zeros elsewhere. We call \( \mathbf{Q} \in \mathbb{C}^{p \times q} \) a left-\( \mathbf{P} \)-real matrix if it satisfies \( \mathbf{P}_p \mathbf{Q}^* = \mathbf{Q} \). The unitary matrix

\[
\mathbf{Q}_{2n+1} = \frac{1}{\sqrt{2}} \begin{bmatrix}
I_n & 0 & jI_n \\
0 & \sqrt{2} & 0 \\
I_n & 0 & -jI_n
\end{bmatrix}
\]

(1)
is left-\( \mathbf{P} \)-real of odd order. A unitary left-\( \mathbf{P} \)-real matrix of size \( 2n \times 2n \) is obtained from \( \mathbf{Q}_{2n+1} \) by dropping its center row and center column.

Let \( (\cdot)^H \), \( (\cdot)^T \), \( (\cdot)^* \), and \( E\{\cdot\} \) denote the Hermitian transpose, the transpose, the conjugation, and the expectation operator, respectively. Column vectors and matrices are denoted by bold lowercase \( (a, b, \ldots) \) and bold uppercase letters \( (A, B, \ldots) \), respectively.

Definition

A sensor array comprising \( M \) sensors is called centro-symmetric if its element locations are symmetric with respect to the centroid and the complex radiation characteristics of paired elements are the same. Their array steering matrix \( \mathbf{A} \), therefore, satisfies \( \mathbf{P}_M \mathbf{A}^T = \mathbf{A} \) for any unitary diagonal matrix \( \mathbf{A} \) [6]. A uniform linear array (ULA) is a centro-symmetric array.

II. DATA MODEL

Assume a ULA of \( M \) sensors (or antennas) in the far field of \( d \) sources. A single snapshot of the noise corrupted data at the output of the \( M \) sensors is given by

\[
x = \mathbf{A}s + \mathbf{w}, \quad \in \mathbb{C}^{M \times 1}.
\]

The entries of noise vector \( \mathbf{w} \) are assumed to be i.i.d., zero-mean circularly symmetric complex Gaussian distributed with variance \( \sigma^2 \). Let \( \Delta \) denote the distance between two adjacent sensors, \( \theta_k \) denote the DOA of the \( k \)th source, \( \lambda \) denote the

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wavelength of the impinging wavefront, and \( s \in \mathbb{C}^{d \times 1} \) denote the complex signal vector at the array due to the \( d \) sources. We study the undamped case and, thus, the array steering matrix \( A \) is given by

\[
A = \begin{bmatrix}
    \Phi^T \\
    \Phi^T \\
    \ldots \\
    \Phi^T \\
    \Phi^T M^{-1}
\end{bmatrix}
\]

where

\[
\Phi = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_d}), \quad \Phi^T = [1, 1, \ldots, 1]
\]

and the \( k \)th spatial frequency is \( \mu_k = -(2\pi/\lambda)\Delta \sin \theta_k \). Our aim is to estimate \( \{\theta_k\}_{k=1}^d \).

III. SPATIAL SMOOTHING

Spatial smoothing is applied to \( \Phi \) in (2) to get a measurement matrix \( X_{ss} \) that has a rank greater than or equal to \( d \), while \( d \) is assumed to be known. For \( M = M_{\text{sub}} + L - 1 \), we define

\[
J_j^{(M)} = [0_{M_{\text{sub}} \times (L-1)} \ I_{M_{\text{sub}}} \ 0_{M_{\text{sub}} \times (L-1)}], \quad 1 \leq l \leq L,
\]

\[
X_{ss} = [J_1^{(M)} x \ J_2^{(M)} x \ldots J_L^{(M)} x] \in \mathbb{C}^{M_{\text{sub}} \times L}.
\]

Then

\[
X_{ss} = A_s B_s + W_s
\]

where

\[
B_s = [\Phi_0, \Phi_0^2, \ldots, \Phi_0^{L-1}] \in \mathbb{C}^{d \times L}, \quad A_s = \text{diag}(s).
\]

The new array steering matrix

\[
A_s = [I_{M_{\text{sub}}} \ 0_{M_{\text{sub}} \times (M-M_{\text{sub}})}] A
\]

is the steering matrix of a centro-symmetric array. Each of the columns of \( W_s \) is white, i.e., \( E\{W_s(j)W_s(j)^H\} = \sigma^2 I_{M_{\text{sub}}} \) for \( j = 1, 2, \ldots, L \), but the columns of \( W_s \) are mutually correlated. However, subspace-based algorithms like MUSIC [7], Matrix-Pencil, ESPRIT, and Unitary ESPRIT can be applied to (5) to estimate the DOAs \( \{\theta_k\}_{k=1}^d \). It is to be noted that the effective aperture, i.e., the effective number of sensors reduces from \( M \) to \( M_{\text{sub}} \) after the conventional spatial smoothing.

IV. SPATIAL SMOOTHING WITH IMPROVED APERTURE

Observe that both \( A_s \) and \( B_s^T \) in (5) are steering matrices of centro-symmetric arrays. Therefore

\[
\Pi_{M_{\text{sub}}} A_s^T = A_s A_1 \quad \text{and} \quad \Pi_L B_s^T = B_s^T A_2
\]

where \( A_1, A_2 \) are unitary diagonal matrices of size \( d \times d \). A similar observation has been reported in [8], but the authors used only the structure of \( A_s \) to estimate \( \{\theta_k\}_{k=1}^d \).

We exploit the structures of both \( A_s \) and \( B_s^T \) and construct an augmented matrix

\[
X_{\text{aug}} = \begin{bmatrix}
    X_{ss} \\
    \Pi_{M_{\text{sub}}} X_{ss} \Pi_L
\end{bmatrix}
\]

Here, \( A_{\text{aug}} \in \mathbb{C}^{2M_{\text{sub}} \times d} \) is the steering matrix of a centro-symmetric array of size \( 2M_{\text{sub}} \). In other words, we have created a centro-symmetric array that has twice the number of sensors than (5). The number of rows in the augmented steering matrix, \( A_{\text{aug}} \), is twice the number of rows in the steering matrix \( A_s \). Each of the columns of the noise matrix \( W_{\text{aug}} \) is white with

\[
E\{W_{\text{aug}}(j)W_{\text{aug}}(j)^H\} = \sigma^2 I_{2M_{\text{sub}}} \quad \text{for} \quad j = 1, 2, \ldots, L, \quad \text{but the columns of} \quad W_{\text{aug}} \quad \text{are mutually correlated.}
\]

The augmented array steering matrix \( A_{\text{aug}} \) satisfies the following shift invariance property

\[
J_j^{(1)} A_{\text{aug}} \Phi = J_j^{(2)} A_{\text{aug}} \Phi
\]

If we construct two subarrays with maximum overlap from the centro-symmetric sensor array of size \( 2M_{\text{sub}} \), then

\[
J_j^{(0)} = I_2 \otimes J_j, \quad j = 1, 2
\]

where \( m = (M_{\text{sub}} - 1) \), and

\[
J_1 = [I_m \quad 0_{m \times 1}], \quad J_2 = [0_{m \times 1} \quad I_m].
\]

The unknown array steering matrix \( A_{\text{aug}} \) is replaced by the matrix \( U_{\text{aug}} \), that contains the \( d \) dominant left singular vectors of the matrix \( X_{\text{aug}} \), in the following way:

\[
J_j^{(1)} U_{\text{aug}} \Psi = J_j^{(2)} U_{\text{aug}} \Psi
\]

where \( \{\mu_k\}_{k=1}^d = \text{arg}\{\text{eig}(\Psi)\} \).

DOA estimation algorithms that are based on the shift invariance property of the array steering matrix \( A_{\text{aug}} \), for example, Matrix-Pencil, ESPRIT, and Unitary ESPRIT can be applied to (6). The knowledge of the symbols, \( s \), would be necessary for applying MUSIC to (6). In this paper, we apply Unitary ESPRIT to (6) as described below.

Note that \( X_{\text{aug}} \) is centro-Hermitian and satisfies the following identity:

\[
\Pi_{2M_{\text{sub}}} X_{\text{aug}} \Pi_L = X_{\text{aug}}.
\]

Therefore, \( X_{\text{aug}} \) can be transformed to a real valued matrix according to [2]

\[
\varphi(X_{\text{aug}}) = Q_{\text{L}}^H X_{\text{aug}} Q_{\text{L}}
\]

where \( Q_{2M_{\text{sub}}} \) and \( Q_{\text{L}} \) are left-\( \Pi \)-real matrices. It can be shown that for an even \( L \), \( X_{ss} \) can be partitioned as

\[
X_{ss} = [U \ V], \quad U, V \in \mathbb{C}^{M_{\text{sub}} \times L/2}
\]

\[
\varphi(X_{\text{aug}}) = \begin{bmatrix}
    \text{Re}[U] + \text{Re}[V] \Pi_{L/2} & \text{Im}[U] - \text{Im}[V] \Pi_{L/2} \\
    \text{Im}[U] + \text{Im}[V] \Pi_{L/2} & \text{Re}[U] - \text{Re}[V] \Pi_{L/2}
\end{bmatrix}
\]

if the left-\( \Pi \)-real matrices \( Q_{2M_{\text{sub}}} \) and \( Q_{\text{L}} \) are constructed according to (1). Thus, the transformation \( \varphi(X_{\text{aug}}) \) can be computed using only 2LM_{\text{sub}} real additions. For an odd \( L \),
$(2L - 1)M_{sub}$ real additions and $M_{sub}$ real multiplications are needed to compute $\varphi(X_{aug})$. The overdetermined set of complex-valued linear equations given by (9) can be converted to the following overdetermined set of real-valued linear equations:

$$K_{aug}^{(1)} = K_{aug}^{(2)} \approx K_{aug}^{(2)} E_{aug} X_{aug} \gamma_{aug}$$

(12)

where $E_{aug}$ contains the $d$ dominant left singular vectors of $\varphi(X_{aug})$. Moreover, $K_{aug}^{(1)}$ and $K_{aug}^{(2)}$ are the transformed selection matrices. As in the traditional version of Unitary ESPRIT, $\gamma_{aug} = \text{diag} \{ \tan(\mu_k/2) \}_{k=1}^d$ are related through a similarity transformation [2]. The transformed selection matrices are obtained from $J_{aug}$ in the following way:

$$K_{aug}^{(1)} = 2 \cdot \text{Re} \left\{ Q^H 2m_{aug} J_{aug}^2 Q 2M_{sub} \right\}$$

and

$$K_{aug}^{(2)} = 2 \cdot \text{Im} \left\{ Q^H 2m_{aug} J_{aug}^2 Q 2M_{sub} \right\}.$$ 

Least squares (LS), structured least squares (SLS) [9], or total least squares (TLS) can be used to solve (12) for $J_{aug}$ in the real domain. If all eigenvalues of $\gamma_{aug}$ are real, we call the estimates reliable[2]. If the eigenvalues of $\gamma_{aug}$ occur in complex conjugate pairs, then the corresponding phase factors $e^{j\mu_k}$ will not be on the unit circle. Therefore, if the eigenvalues of $\gamma_{aug}$ are not real, the Unitary ESPRIT reliability test has failed, and the algorithm has to be started with a new set of measurements. Finally the spatial frequencies are given by $\{ \mu_k \}_{k=1}^d = 2 \pi \text{arctan} (\text{sgn}(\gamma_{aug}))$.

We need $(M_{sub} - d)$ for estimating $d$ sources from $X_{aug}$ considering maximum overlap between the subarrays. Again, the condition $L \geq d$ must be satisfied. In other words, $(M_{sub})_{\text{min}} = (d + 2)/2$ and $(L)_{\text{min}} = d$. Thus, $(M_{sub})_{\text{min}} = (M_{sub})_{\text{min}} + (L)_{\text{min}} - 1 = 3d/2$, i.e., at least $3d/2$ sensors are needed to detect $d$ sources, or for a given $M$, the proposed method can determine DOAs of up to $(2/3)M$ sources using a single snapshot.

We do not apply forward-backward averaging to the matrix $X_{aug}$ as such a step does not increase the number of data samples because of identity (10).

V. SIMULATION RESULTS

For the simulation purpose, we assume $\Delta = \lambda/2$. Thus, $\mu_k = -\pi \sin \theta_k$. We solve (12) using LS. Moreover, we choose $M_{sub}$ and $L$ such that $X_{ss}$ becomes a square matrix (odd $M$) or an almost square matrix (even $M$). This choice stems from the fact that the left-subspace and the right-subspace of a square $X_{ss}$ are equally good. An alternative choice for $M_{sub}$ could be $M_{sub} = \lfloor (2d/3)M \rfloor$ so that after forward-backward averaging the measurement matrix becomes square. We refer to this tall matrix with $M_{sub} = \lfloor (2d/3)M \rfloor$ as $X_{alternate}$. (Unitary ESPRIT incorporates forward-backward averaging by transforming a complex data matrix into a real one). When a user is unaware of the structure of $X_{ss}$, as explained in Section IV, $X_{alternate}$ is the best choice for the DOA estimation for a given $d$ and $M$. SSIA in conjunction with Unitary ESPRIT is compared with 1). ESPRIT applied to $X_{ss}$, 2). Unitary ESPRIT applied to $X_{ss}$, and 3). Unitary ESPRIT applied to $X_{alternate}$ in terms of the root mean square error (RMSE) defined as

$$\text{RMSE} = \sqrt{E \left\{ \sum_{k=1}^{d} (\mu_k - \hat{\mu}_k)^2 \right\}}.$$ 

The estimated spatial frequency is $\hat{\mu}_k$. The sizes of the matrices for all the four cases are indicated as $r \times c$ in the figures, where $r$ denotes the number of rows and $c$ is the number of columns. The deterministic Cramér-Rao lower bound (CRB) for the spatial frequency estimation is also plotted in all the figures [7].

Fig. 1 depicts the RMSE vs. the SNR (signal-to-noise ratio) when a single snapshot from the output of an array of 34 sensors is used to estimate the DOAs of 9 sources that emit QPSK symbols. The nine DOAs are $\theta_1 = -30^\circ$, $\theta_2 = -24^\circ$, $\theta_3 = -18^\circ$, $\theta_4 = -12^\circ$, $\theta_5 = -6^\circ$, $\theta_6 = 0^\circ$, $\theta_7 = 6^\circ$, $\theta_8 = 12^\circ$, $\theta_9 = 18^\circ$. SSIA in conjunction with Unitary ESPRIT is significantly better than ESPRIT applied to $X_{ss}$ and Unitary ESPRIT applied to $X_{ss}$ at all SNRs. At a RMSE of $10^{-2}$, Unitary ESPRIT applied to $X_{ss}$ and ESPRIT applied to $X_{ss}$ are, respectively, 6 and 10 dB worse than SSIA in conjunction with Unitary ESPRIT. SSIA in conjunction with Unitary ESPRIT gives a significantly better performance than Unitary ESPRIT applied to $X_{alternate}$ at SNRs below 20 dB.

Fig. 2 shows the failure rate of the reliability test for 34 sensors in % for: a) SSIA in conjunction with Unitary ESPRIT; b) Unitary ESPRIT applied to $X_{ss}$; and c) Unitary ESPRIT applied to $X_{alternate}$. SSIA does not fail at SNRs above 5 dB and it provides the smallest failure rate above 0 dB SNR.

Fig. 3 shows the change in RMSE as a function of the SNR for 34 sensors and 9 sources with DOAs $\theta_1 = -30^\circ$, $\theta_2 = -24^\circ$, $\theta_3 = -18^\circ$, $\theta_4 = -12^\circ$, $\theta_5 = -6^\circ$, $\theta_6 = 0^\circ$, $\theta_7 = 6^\circ$, $\theta_8 = 12^\circ$, $\theta_9 = 18^\circ$. The sources emit complex Gaussian symbols. SSIA in conjunction with Unitary ESPRIT is better than the other three methods at all SNRs. Unitary ESPRIT applied to $X_{ss}$ comes closest to SSIA in conjunction with Unitary ESPRIT and is on the average 20% worse than SSIA in conjunction with Unitary ESPRIT at all SNRs.
VI. CONCLUSION

SSIA is a preprocessing technique that improves the effective array aperture significantly as opposed to the conventional spatial smoothing by doubling the rows of the array steering matrix. Therefore SSIA in conjunction with subspace-based DOA estimation technique performs better than the existing preprocessing techniques, such as spatial smoothing and forward-backward averaging, in conjunction with subspace-based DOA estimation technique at all SNRs and for all values of the number of sensors, the number of sources, the DOAs, and the separation between the sources. The threshold behavior (at small SNRs) is significantly improved. Also note that SSIA works for an arbitrary signal constellation.

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REFERENCES


