A Root-CMA Algorithm for Multi-User Separation

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Abstract—Constant modulus algorithms often offer candidate solutions for problems related to wireless communication systems that use antenna arrays and also for MIMO radar. The classical constant modulus adaptive array, however, fails at locking onto a single mode, and instead, it equalizes the full band spectrum. In this paper, we develop a novel approach for the separation of multiple users in a frequency-reuse radio system making use of the original constant modulus adaptive (CMA) algorithm. Based on the observation that the differential filter weights resemble a superposition of the array steering vectors, we cast the original task to a direction-of-arrival estimation problem. With rigorous theoretical analysis of the array response based on the discrete-space Fourier transform we elaborate a solution that solves the problem by finding the roots of a polynomial equation. We give an illustrative numerical example to demonstrate the validity of the approach under high SNR conditions.

Index Terms—adaptive and array signal processing, constant modulus modulation, blind multi-user separation, discrete-space Fourier transform, polynomial roots

I. INTRODUCTION

A constant modulus adaptive (CMA) algorithm represents a class of adaptive digital filtering algorithms that are sensitive to amplitude modulation and insensitive to angle modulation. Most commonly, it is realized as a transversal finite-impulse-response (FIR) digital filter based on gradient descent. In its original formulation the CMA algorithm compensates for the frequency-selective multipath and interference on signals that have a constant envelope. Such radio signals can be generated using frequency modulation or quadrature phase-shift keying (QPSK) [1]. Despite closely resembling the well-known least mean squares (LMS) filter, the main feature of a CMA filter is that it does not require a reference signal to operate.

According to the authors of [1], the CMA filter is not only useful to counter multipath induced fading, but also to reduce narrow-band or sinusoidal interference if the interferer’s power is far below carrier power (–10 dB or less). In that case, the CMA filter acts as a notch filter. Nevertheless, it is pointed out that extra measures, such as prefiltering, are necessary in order to deal with a strong interferer. A more detailed analysis of the phenomenon is carried out in [2]. There, based on the model of two sinusoids as the input, the authors demonstrate that if the signals have nonoverlapping spectra, it is possible to find two different types of solutions for the filter: one to suppress the interferer and to capture the signal of interest, but all the same it may happen that the interferer is taken for the signal of interest and is captured instead.

With regard to the convergence of the CMA array we find works such as [3] or also [4]. Both rely on the capture effect of the CMA filter, i.e. the assumption that the algorithm will lock onto one single CM signal with a particular polarization or direction of arrival, in case it is the strongest one present. In [5], however, it is pointed out that even if the two sources are separated by 90°, i.e. their directions are perpendicular to each other, the array’s capture effect is difficult to predict, as the filter may switch arbitrarily from one source to the other. The problem of capturing one particular source in the presence of multiple CM sources is acknowledged in [6]. Yet, it is wrongly stated that the problem can be solved in multiple stages. So, if the CMA array fails lock, it cannot converge, see Fig. 1.

The family of in a broader sense called “constant modulus” algorithms is widely used in wireless communication systems and in radar. In [7], e.g., a CM algorithm serves as the basis for low-complexity semi-blind adaptive beamforming in an array-based communication system. In [8], on the other hand, a CM algorithm is employed to estimate the carrier frequency offset in an optical system. There again, a CM algorithm is used in [9] to design the probing waveform for a MIMO radar.

In this paper, we develop a novel approach for the multi-user problem using the CMA array. Since our approach does not rely on any sort of capture effect, it is, to the best of our knowledge, the only CMA-based algorithm that is applicable to the problem. The analytical CMA (ACMA) [10]–[12] is the other alternative.

II. SIGNAL MODEL AND PROBLEM STATEMENT

Consider the classical signal model

$$X = AS^H + N,$$ (1)

where $S \in \mathbb{C}^{N \times D}$ is the signal matrix,

$$S = [s_1 \ s_2 \ \cdots \ s_D]$$ (2)

with the $d$th column belonging to the $d$th source signal, $A \in \mathbb{C}^{M \times D}$ is the array response matrix, the columns of which are vectors on the array manifold associated with a direction of arrival (DOA) or spatial frequency,

$$A = [a_1 \ a_2 \ \cdots \ a_D],$$ (3)

$N \in \mathbb{C}^{M \times N}$ is a (zero-mean) Gaussian noise matrix, and $X \in \mathbb{C}^{M \times N}$ contains $N$ temporal snapshots collected by $M$ antennas. On the assumption that the antennas are arranged
as a uniform linear array (ULA), A becomes a Vandermonde matrix, i.e.

$$a_d = \begin{bmatrix} 1 & z_{d} & z_{d}^2 & \cdots & z_{d}^{M-1} \end{bmatrix}^T$$

with $z_{d} = e^{j2\pi \xi_d}$, (4)

where $e$ is Euler’s number, $j$ is the imaginary unit, and $\xi_d$ is the spatial frequency shift,

$$\xi_d = \frac{\Delta}{\lambda} \sin \theta_d, \quad (5)$$

where $\Delta$ is the spacing between the elements of the antenna, $\lambda$ is the wavelength and $\theta_d$ denotes the inclination angle of the $d$th wavefront.

The noise matrix $N$ put aside, the problem at hand can be formulated as a structured matrix factorization problem:

$$X \simeq \hat{A}\hat{S}^H \quad \text{s.t. rank} \ A = \text{rank} \ \hat{S} = D \quad \text{and} \ |\hat{s}_{n,d}| = 1. \quad (6)$$

The task is to find the factors $A$ and $\hat{S}$ for a given $X$, with $\hat{S}$ satisfying the constant modulus property.

### III. Constant-Modulus Adaptive Algorithm

The original single-channel CMA filter is discussed at full length in [1]. In the following, we give a summary of how to design it. First, consider a sampled quadrature signal $x(n)$ to be frequency modulated and multipath distorted. This signal, which is analytic and hence complex, passes through a tapped delay line with adjustable complex weights, which represents the transversal FIR filter with filter coefficients $w$. Then, the complex filter output may be written as

$$y(n) = w^H(n) x(n), \quad (7)$$

where

$$x(n) = \begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-N+1) \end{bmatrix}^T \quad (8)$$

is the data in the delay line at sampling instant $n$ and

$$w(n) = \begin{bmatrix} w_0(n) & w_1(n) & \cdots & w_{N-1}(n) \end{bmatrix}^T \quad (9)$$

is the vector of $N$ adjustable filter coefficients. Conveniently, we assume that the coefficients are tweaked for each $n$.

The objective now is to restore the output $y(n)$ to a form, which on average has a constant instantaneous modulus. It is achieved by choosing the coefficients $w$ in such a manner as to minimize the following cost or performance function:

$$J = \frac{1}{4} E \left\{ \left| y(n) \right|^2 - 1 \right|^2 \right\}, \quad (10)$$

where $E$ denotes expectation.

With the cost function and the filter structure given, (10) is minimized using stochastic gradient descent. In particular, $w$ is updated according to the recurrence relation

$$w(n+1) = w(n) - \gamma \nabla_w J(n), \quad (11)$$

where $\gamma$ is the step size (adaptation constant) and $\nabla_w$ is the gradient with respect to the filter coefficients. Approximating the stochastic gradient

$$\nabla_w J(n) = E \left\{ \left[ \left| y(n) \right|^2 - 1 \right]\cdot x(n)y^*(n) \right\} \quad (12)$$

by the instantaneous gradient leaving out the expectation, the update rule from (11) writes

$$w(n+1) = w(n) - \gamma \left[ \left| y(n) \right|^2 - 1 \right]\cdot x(n)y^*(n). \quad (13)$$

A multi-channel spatial filter can be derived easily from the transversal filter. Changing the independent time variable $n$ in (8) to the channel index $m$ indicating an antenna element, we instantly obtain the formulation of a CMA array that can be put into effect in an adaptive antenna framework [5].
IV. DISCRETE-SPACE FOURIER TRANSFORM

Before going into detail about how to solve the multi-user separation problem using the output from the original CMA, let us first introduce the discrete-space Fourier transform. The latter, together with the z-transform, are resorted to in Section V, so as to have a mathematical basis for the pursued approach and to justify the results.

A. Definition

The discrete-space Fourier transform (DSFT) of a series of real (or complex) numbers is a Fourier series that is periodic w.r.t. the angular spatial frequency variable $\mu$, also known as the angular wave number. The DSFT is defined as:

$$X(e^{j\mu}) = \sum_{m=-\infty}^{\infty} x(m) e^{-j\mu m}. \quad (14)$$

The corresponding inverse is given by

$$x(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\nu}) e^{j\mu m} d\nu. \quad (15)$$

B. Transform of a Finite-Length Sequence

The expression in (14) applies to an infinite series $x(m)$. In order to analytically evaluate the DSFT of a finite-length data sequence, we apply a rectangular window of length $M$ to the input sequence $x(m)$, resulting in:

$$X(e^{j\mu}) = \sum_{m=0}^{M-1} x(m) e^{-j\mu m}. \quad (16)$$

Using the convolution theorem, it can be shown that the DSFT of a finite-length sequence as in (16) is identical with

$$\hat{X}(e^{j\mu}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-j\mu \frac{m-1}{2}} \sin(M\frac{\nu}{2})}{\sin(\frac{\nu}{2})} X[e^{j(\mu-\nu)}] d\nu, \quad (17)$$

see also [13], [14].

C. Fourier Series and Array Response

Consider the case where we seek to compute the DSFT of the array steering vector $a_m = e^{j\mu_0 m}$, $m = 0, 1, \ldots, M - 1$, which can be written as $a(m) = e^{j\mu_0 m} x(m)$ with $x(m) = 1$. One can easily show that, for $-\infty < m \leq \infty$,

$$\mathcal{F}\{a(m)\} = A(e^{j\mu}) = 2\pi \sum_{\kappa=-\infty}^{\infty} \delta(\mu - \mu_0 - 2\pi\kappa). \quad (18)$$

where $\mathcal{F}$ denotes the DSFT. Thus, for $m = 0, 1, \ldots, M - 1$,

$$A(e^{j\mu}) = e^{-j(\mu-\mu_0)\frac{m-1}{M-1}} \sin(M\frac{\mu-\mu_0}{2}) \sin(\frac{\mu-\mu_0}{2}). \quad (19)$$

To derive (19), we use the sifting property of the Dirac delta function. From (19) we hence conclude that the DSFT of the array steering vector is given by the (causal) Dirichlet kernel shifted by $\mu_0$ from zero. It has a global maximum at $\mu_0$ with a value of

$$A_{\text{max}}(e^{j\mu}) = \lim_{\mu \to \mu_0} A(e^{j\mu}) = M. \quad (20)$$

The relation to the array response is established by taking a closer look at its definition. It is very common to write the output as in (7). Hence, the array or beam response $b(\mu)$ can be formulated as

$$b(\mu) = w^H a(\mu), \quad (21)$$

which is a function of the direction and the weights. Now, if we plug the steering vector $a_m$ into (21), it reads

$$b(\mu) = \sum_{m=0}^{M-1} w^* e^{j\mu m}. \quad (22)$$

Then, comparing (22) with (16), we see that

$$b(\mu) = W^* (e^{j\mu}), \quad (23)$$

where $W(e^{j\mu})$ is the DSFT of the weights.

D. Transform of a Sum of Steering Vectors

We close this part with a brief analysis of the special case where the space sequence is generated by superposition of a number of distinct array vectors, i.e.

$$a(m) = \sum_{d=1}^{D} a_{dm} = \sum_{d=1}^{D} e^{j\mu_0 d m}, \quad (24)$$

for $m = 0, 1, \ldots, M - 1$. Since the DSFT is linear, it follows that the DSFT of a sum of finite space sequences is equal to the sum of DSFTs of the sequences alone. And therefore,

$$A(e^{j\mu})|_{\mu=\mu_i} = \frac{1}{M} 
+ \sum_{d=1}^{D} \sum_{d \neq i} \re^{j(\mu_i-\mu_d) \frac{m-1}{M-1}} \sin(M\frac{\mu_i-\mu_d}{2}) \sin(\frac{\mu_i-\mu_d}{2}), \quad (25)$$

where $A(e^{j\mu_0}) \neq M$, in general, because of intermodulation products. Yet, if the difference $|\mu_i - \mu_d|$ is equal to $k \cdot \frac{2\pi}{M-1}$, $k = 1, 2, \ldots$, the phase term is $\pm 1$, and (25) is purely real. Then, $A(e^{j\mu_0})$ has a value that can be computed as

$$A(e^{j\mu_0})|_{\mu_i=\mu_d\pm k \frac{2\pi}{M-1}} = M + D - 1. \quad (26)$$

V. MULTIUSER SEPARATION

In this section, we elaborate our proposed extension of the CMA algorithm which we call the root-CMA algorithm.

A. Exploratory Experiment

Consider three QPSK sources being located in the far field of a receiver in a radio system with frequency reuse. Further assume that the point sources are at equal distance, such that the incoming signals have the same amplitude. The angles of inclination of the three plane waves should obey (26), i.e.

$$\theta_d = \arcsin \left[ \sin(\theta_t \pm \frac{\lambda}{\Delta(M-1)} \right]. \quad (27)$$

No explicit orthogonality scheme such as polarization should be in use. In Fig. 2, by way of experiment, we show that the flipped, cleansed, and normalized CMA array response is, in good approximation, equal to the sum of the steering vectors.
Fig. 2. Inverted, cleansed, and normalized magnitude response of the CMA array for $D = 3$ distinct sources in the case where the phase relation holds ($M = 8$, $\theta_1 = -53.2^\circ$, $\theta_2 = 3.25^\circ$, $\theta_3 = 20.0^\circ$, SNR = 20 dB).

associated with the point sources (see also Fig. 1). The array response is flipped by inverting the direction of the gradient, converting (11) into a gradient ascent algorithm. The weight vector $w(n)$ is then cleansed by subtracting the initial vector $w_0$ and finally normalized according to

$$v(n) = \frac{w(n) - w_0}{\|w(n) - w_0\|_2} \sqrt{D^2 + D(M - 1)}. \quad (28)$$

B. Problem Restatement

On the hypothesis that $v(n)$ is the sum of steering vectors, the multi-user separation problem is recast as follows. Given $v(n)$, find $\{\hat{a}_d\}_D$ that satisfies

$$v^H(n) \hat{a} \simeq \|\hat{a}\|^2_2 \quad \text{with } \hat{a} = \sum_{d=1}^D \hat{a}_d, \quad (29)$$

or, in terms of the DSFT, find

$$\{\hat{\mu}_d \in [-\pi, \pi) : |V^*(e^{j\hat{\mu}_d}) - C| < \varepsilon\} \quad \text{with } \varepsilon \to 0, \quad (30)$$

for $d = 1, 2, \ldots, D$, where $C = M + D - 1$. I.e., we seek after $D$ unique frequencies for the steering vectors, the DSFTs of which (in good approximation) sum up to the DSFT of $v(n)$.

C. Basic Approach

Substituting $e^{j\hat{\mu}} = z$, we postulate that

$$V^*(z) \simeq C. \quad (31)$$

Converting (31) to a polynomial and normalizing it by $v^*_{M-1}$, we obtain

$$P(z) = \frac{v^*_0 - C}{v^*_{M-1}} + \sum_{m=1}^{M-1} \frac{v^*_m}{v^*_{M-1}} z^m. \quad (32)$$

The product representation of (32) is

$$P(z) = \prod_{m=1}^{M-1} (z - z_m), \quad (33)$$

where $z_m$ are the roots of $P(z)$. The sought-after frequencies are the arguments of the $D$ roots that lie the closest to the unit circle, see Fig. 3.

VI. CONCLUSION

We presented a novel approach for the separation of multiple sources in a frequency-reuse radio system making use of the classical CMA array. Our extension of the CMA array makes it applicable when one particular source is to be captured in the presence of multiple CM sources. The literature cited relies on an implicit capture effect of the algorithm, which is hardly sustainable looking at the formulation of the underlying cost function. The CMA array rather acts as a soft spatial equalizer. This fact can be exploited to reconstruct the array response matrix and to find the best fit for the constant modulus signals in, e.g., the least-squares sense immediately after.

REFERENCES


